Binary Constraint Satisfaction Games

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Abstract

This work investigates the idea of using quantum entanglement in reference to non-local games. We begin with the definition of a non-local game and study it's importance in context to the higher success probability gained over playing the game using optimal classical strategies. For certain non-local games, like the XOR games, the gap between the value of the game(maximum success probability) played classically and other using entanglement sharing can be computed efficiently. But for a general class of non-local games, while determining the classical value of the game is NP-Hard, the problem of computing the quantum value is not known to be decidable. Even deciding whether there is a perfect quantum strategy is not known to be decidable. We consider a special class of non-local games, namely the binary constraint system (BCS) games, and look whether they admit a perfect strategy or not. Next, we take a BCS game and give a quantum strategy which has a success probability higher than that of the classical value of that game. We remark that this gap in the winning probability for quantum and classical strategies can be linked to contextuality.

1 Introduction

A non-local game consists of two players (Alice, Bob) who cooperate with each other to jointly win the game. A referee runs the game and all the communication during the game is between the referee and players- no inter player communication is allowed once the game starts. The referee has his distribution of questions, and randomly selects one question for each players and sends each question to the appropriate player. Now, the players send back answers to the referee and based on the answers and questions, the referee decides whether the players win or lose the game. The referee's distribution of the questions and the winning predicate is publicly known to the players.

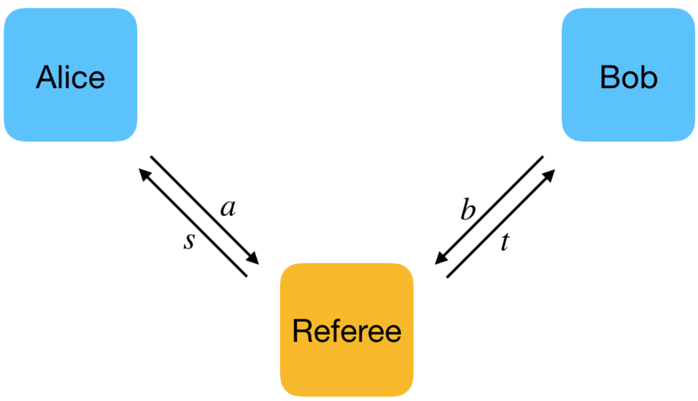

Figure 1: Figure illustrating a non-local game with two players

1.1 Game Definition

A non-local game is defined as G = G(Π,V), where π is the probability distribution on the question set $S \times T$ of referee and the V is the predicate on $S \times T \times A \times B$, such that V is 1 when both the players answer the question correctly and 0 otherwise. The referee chooses the questions (*s,t*) $\in S \times T$ randomly from the distribution Π, and sends *s* to Alice and *t* to Bob. Alice then replies with $a \in A$ and similarly Bob replies with $b \in B$. Note that upon receiving the questions, the players are not allowed to communicate with each other. The predicate V on (*s,t,a,b*) is written as V$(a, b \mid s, t)$ which gives a value when, given *s, t*, the players output *a, b* respectively.

1.2 Game Value

A game value is defined as the maximum success probability achievable by the players. Now there are two ways for the players to play the game:

1. Either they choose to share a quantum state, in which case the game value is denoted by ω_c.
2. Or, they do not share a quantum state and try to work out the best possible classical strategy beforehand, in which case the game value is denoted by ω_q.

The *classical* value of the game is defined as follows:

$$\omega_c(G(V,\Pi)) = \max_{a,b} \sum_{s,t} \Pi(s,t)V(a, b \mid s, t) \tag{1}$$

The maximization is over all possible strategies including the probabilistic strategies. However a probabilistic strategy value can be written as convex combination of the deterministic strategies. Hence the maximum value of the game can be achieved using a deterministic strategy. Hence from here, $a \to a(s)$ and $b \to b(t)$.

A *quantum* strategy is defined by a shared entangled state $|\psi\rangle \in \mathbb{C}^{d\times d}$ for $d \geqslant 1$ between the players Alice and Bob who use projection operators to measure their part of the quantum state. The table below describes a quantum strategy for Alice and Bob:

Alice	Bob
$\{A_a^s\}$, ∀ question s	$\{B_b^t\}$, ∀ question t
$A_a^{s\dagger} = A_a^s$	$B_b^{t\dagger} = B_b^t$
$\sum_a A_a^s = I$	$\sum_b B_b^t = I$

The probability that on questions *s,t* Alice and Bob answer *a,b* respectively is given by $\langle\psi|A_a^s \otimes B_b^t|\psi\rangle$. The *entangled* value of the game ω_q is given by:

$$\omega_q(G(V,\Pi)) = \lim_{d\to\infty} \max_{|\psi\rangle, A_a^s, B_b^t} \sum_{a,b,s,t} \Pi(s,t)V(a, b \mid s, t)\langle\psi|A_a^s \otimes B_b^t|\psi\rangle \tag{2}$$

Note that instead of projection operators, we could have started with a much more generalized POVM operators, but any POVM operators can be expressed as a projection operator in a higher dimensional Hilbert space and since here we are not restricting the dimensionality of the Hilbert space, hence there is no loss of generality in assuming them to be projection operators.

2 Bell's Inequality

In 1935, Einstein, Podolsky and Rosen, developed a thought experiment, they called it the EPR Paradox where they imagined two initially interacting physical systems described by a single quantum state $|\psi\rangle$. Once separated, the two systems(lets say photons) are still described by $|\psi\rangle$, and a measurement of an observable in one system instantaneously affects the state of corresponding observable in the second system even though the systems are no longer physically linked and may be light years separated. Several hidden variable theories came up, which claimed to describe such a scenario.

In 1964, Bell proposed a mechanism to look for the hidden variables using his famous Bell's inequality [1]. A violation of these inequalities would lead to failure of such a theory. He introduced entanglement which could perfectly explain the EPR paradox and would lead to the violation of Bell's inequality, thus discarding the concept of hidden variables and introducing a new central feature in QM.

Non Local Games provide a natural setting for describing Bell's inequality. This inequality is analogous to the probability of winning the non-local game using the best possible classical strategy. Its violation would mean winning the non-local game using a quantum strategy whose game value would be higher than the classical game value. Thus a study of gaps in non-local games is of high relevance to quantum information.

Bell's inequality introduces correlations between outputs of two players. As shown in Figure2 for a given set of inputs and outputs, classical correlations form a convex polytope with fixed vertices corresponding to deterministic strategies. Quantum correlations do not have finite end points, hence these strategies are not known to be decidable. Here NL is a set which contains all non-local correlations, including the non-signalling.

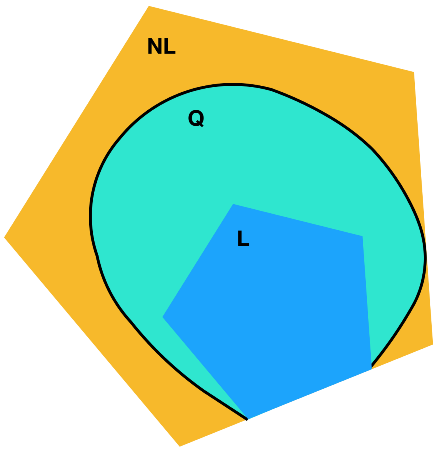

Figure 2: Figure showing the correlations with L being the classical region and admits a finite set of vertices(deterministic strategies), while the Q region has no finite vertices, thus showing that the problem of finding the best strategy is undecidable.NL is global set which contains all correlations.

3 CHSH XOR Game

This is a non-local game involving two players and a referee where the quantum game value can be efficiently determined. This game consists of two players Alice and Bob. The referee's distribution Π is uniform with $st \in \{00, 01, 10, 11\}$. Alice responds to the referee with a single bit a and similarly Bob responds with b. The players jointly win the game if $a \oplus b = s.t$

s	t	$a \oplus b$
0	0	0
0	1	0
1	0	0
1	1	1

For this game, the classical game value $\omega_c = 3/4$, because the players will always answer atleast one question incorrectly. One such strategy is when both the players output 0 in all the cases. However, it can be shown that the quantum game value for this non-local game is $\cos^2(\pi/8) \approx 0.85$.

The quantum strategy is the following. Alice and Bob share a quantum state $|\psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$. The set of operators $\{A_a^s\}$ and $\{B_b^t\}$ for Alice and Bob are defined as follows:

$$\begin{aligned} A_a^0 &= |\phi_a(0)\rangle \langle\phi_a(0)|, \\ A_a^1 &= |\phi_a(\pi/4)\rangle \langle\phi_a(\pi/4)|, \\ B_b^0 &= |\phi_b(\pi/8)\rangle \langle\phi_b(\pi/8)|, \\ B_b^1 &= |\phi_b(-\pi/8)\rangle \langle\phi_b(-\pi/8)| \end{aligned} \tag{3}$$

where,

$$\begin{aligned} |\phi_0(\theta)\rangle &= \cos\theta\,|0\rangle + \sin\theta\,|1\rangle, \\ |\phi_1(\theta)\rangle &= -\sin\theta\,|0\rangle + \cos\theta\,|1\rangle, \end{aligned} \tag{4}$$

So, the upon receiving 0, Alice operates on her part of the shared state with $A^0 = A_0^0 - A_1^0$, and with $A^1 = A_0^1 - A_1^1$ on getting 1. Similarly, Bob has operators $B^0 = B_0^0 - B_1^0$ and $B^1 = B_0^1 - B_1^1$ for inputs 0 and 1 respectively. Now since the probability that on questions s, t Alice answers a and Bob answers b is given by $\langle\phi| A_a^s \otimes B_b^t |\phi\rangle$, hence the maximum success probability can be easily computed for this game and it comes out to be $\approx 85\%$.

4 Binary Constraint System Game

For the XOR game considered previously, the quantum game value is computed efficiently using the semi-definite programming techniques [2, 3]. But for other classes of non-local games, determining the game value is undecidable.

Even determining whether a perfect strategy exists is non known to be decidable as there is no general structure to look for such a strategy. So, instead of looking for a game value (which would be a tougher problem to deal with initially), we would look for some characterization under which a perfect strategy exists. A special class of non-local games i.e. the binary constraint satisfaction game (BCS) is looked at in this section.

4.1 Game Definition

A BCS game consists of n binary variables, $v_1, v_2, ..., v_n$, and m constraints, $c_1, c_2, ..., c_m$, where each c_j is a binary-valued function of a subset of the variables. We take an example of BCS with *n* = 6 and *m* = 4:

$$\begin{aligned} v_1 \oplus v_4 \oplus v_2 &= 0 \\ v_2 \oplus v_5 \oplus v_3 &= 0 \\ v_3 \oplus v_6 \oplus v_1 &= 0 \\ v_4 \oplus v_5 \oplus v_6 &= 1 \end{aligned} \tag{5}$$

In the example shown above, all the constraints are only the functions of parity of variables. Hence, this special BCS is called the *parity* BCS. This above example will be studied in detail in the following section. A BCS is called *satisfiable* if there exists a *truth assignment* to all the variables satisfying all the constraints. It can be seen that the above example provided is *unsatisfiable*.

A BCS game can be seen as a non-local game where the referee sends a constraint c_s to Alice and a variable v_t from the constraint to Bob. Alice then returns the values of variables corresponding to c_s and Bob returns the value for variable v_t. The referee accepts the answers only if,

1. Alice's truth assignment satisfies the constraint c_s.

2. Bob's truth assignment for v_t is consistent with Alice's corresponding variable.

4.2 Satisfying Assignment

To understand Alice and Bob's joint strategy for the BCS game, we define a *quantum satisfying assignment* as the following. First each variable v_j which takes values $\{0, 1\}$ is translated into variables $V_j = (-1)^{v_j}$ that take values $\{+1, -1\}$. Then for any constraint, the parity of a sequence of $\{0, 1\}$ variables is translated into the product of $\{+1, -1\}$ variables. By quantum satisfying assignment, we mean looking for the set of finite dimensional Hermitian operators $A_1, A_2....A_n$ to the variables $V_1, V_2....V_n$ such that,

1. Each A_j is a binary observable such that its eigenvalues are in $\{+1, -1\}$ (i.e.,$A_j^2 = I$).

2. The pair of observable A_i, A_j corresponding to the same constraint should commute. $[A_i, A_j] = 0$.

3. The operators satisfy each constraint $c_s : \{+1, -1\}^k \to \{+1, -1\}$ that acts on variables $V_{i_1}, ..., V_{i_k}$, in the sense that the equation $c_s(A_{i_1}, \cdots, A_{i_k}) = -\mathbb{I}$ is satisfied.

4.3 Perfect Strategy

For classical strategies, determining whether the problem has a perfect strategy is NP-Hard for general BCS games. For parity BCS games though, the constraint equations can be reformulated as linear equations and can be solved in polynomial time. Using entanglement however, there is no characterization to look for perfect strategy.

Cleve & Mittal [4] provide a theorem which relates a perfect strategy with the quantum satisfying assignments used earlier.

Theorem 4.1. *For any binary constraint system, if there exists a perfect quantum strategy for the corresponding BCS game that uses finite or countably-infinite dimensional entanglement, then it has a quantum satisfying assignment*

The detailed proof of the theorem can be found here [4]. Some of the major points outlining the proof are:

1. Assume that there is a perfect entangled strategy of the game and this requires Alice and Bob share an entangled state of the form $|\psi\rangle = \sum_{i=1}^{\infty} \alpha_i |\phi_i\rangle|\psi_i\rangle$, such that $|\phi_i\rangle$ and $|\psi_i\rangle$ are orthonomal states, $\alpha_i \geq 0 \forall i$ and $\sum_{i=1}^{\infty} |\alpha_i|^2 = 1$.

2. Each constraint c_s has k_s variables. The set of outcomes for Alice are then $0, 1^{k_s}$. This can be associated with projection operators Π_s^a. For an observable in the constraint, $A_s^j = \sum_{a \in \{0,1\}^{k_s}} (-1)^{a_j} \Pi_a$.

3. The projection operators $A_1, A_2....A_n$ are constructed for each variable and given a constraint c_s with k_s variables, Alice operates the set observables $A_s^1, \cdots, A_s^k$ corresponding to that constraint on her part of the entangled state. Similarly, when Bob receives a variable v_l from the constraint c_s, he applies the the corresponding observable B_l on his part of the entangled state.

4. Since, the strategy is perfect, Alice's and Bob's measurement outcome on the variable v_l would always agree i.e. $\langle\psi| A_s^j \otimes B_l |\psi\rangle$ = 1 for a variable j in constraint c_s given to Alice and variable v_l on the same constraint given to Bob.

5. Let a same variable be in two constraints c_s and c_s' and is denoted as V_s^j and $V_s^{j'}{}'$ respectively. Now since $\langle\psi| A_s^j \otimes B_l |\psi\rangle = \langle\psi| A_{s'}^{j'} \otimes B_l |\psi\rangle$, this implies $A_s^j = A_{s'}^{j'}$. Hence the observables are non-contextual.

6. Also, since we are dealing with perfect strategy, Alice's output must be consistent with the constraint c_s. That is $\langle\psi|c_s(A_s^1, ..., A_s^j) \otimes B_t|\psi\rangle$ = -1. Hence using the lemma, $c_s(A_s^1, ..., A_s^1) = -\mathbb{I}$.

Note that instead of projection operators, we could have started with a much more generalized POVM operators, but any POVM operators can be expressed as a projection operator in a higher dimensional hilbert space and since here we are not restricting the dimensionality of the hilbert space, hence there is no loss of generality in assuming them to be projection operators. Also, here we started with a mixed entangled state. We could have as well started with a pure entangled state as a mixed state can be expressed as a subset of a pure state of higher dimension.

4.4 Condition for No Perfect Strategy

Speelman [5] showed for some BCS games, a simple technique suffices to show the non existence of a quantum satisfying assignment. Taking the previous example we had considered:

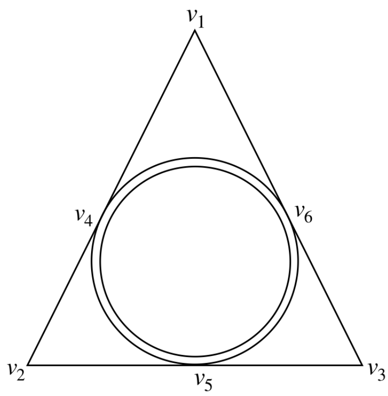

Figure 3: Pictorial representation of a BCS game

$$\begin{aligned} v_1 \oplus v_4 \oplus v_2 &= 0 \\ v_2 \oplus v_5 \oplus v_3 &= 0 \\ v_3 \oplus v_6 \oplus v_1 &= 0 \\ v_4 \oplus v_5 \oplus v_6 &= 1 \end{aligned} \tag{6}$$

We show that this example does not have a quantum satisfying assignment by assuming a contradiction i.e. it has such an assignment. We construct the variables $V_1, V_2, \cdots V_6$ from the original set $v_1, v_2, \cdots, v_6$ and the corresponding observables are $A_1, A_2, \cdots, A_6$. Then:

$$\begin{aligned} A_1A_4A_2 &= \mathbb{I} \text{ (first constraint)} \\ A_1A_4A_5A_6A_1 &= \mathbb{I} \text{ (3rd constraint} A_3 = A_6A_1) \\ A_1A_5A_6A_5A_6A_1 &= \mathbb{I} \text{ (4th constraint} A_4 = A_5A_6) \\ A_1A_6A_5A_5A_6A_1 &= \mathbb{I} \ (A_5A_6 = A_6A_5) \\ \mathbb{I} &= \mathbb{I} \text{ (Hence a contradiction)} \end{aligned} \tag{7}$$

Since this BCS game does not have a quantum satisfying assignment, we will instead try to look for a quantum strategy that wins this BCS game with a probability higher than the classical counterpart.

4.5 Classical strategy and game value

Let the constraints be denoted by c_1, c_2, c_3 and c_4 respectively.

Constraint	Alice's strategy	Bob's strategy
$v_1 \oplus v_4 \oplus v_2$	0 0 0	0 0 0
$v_2 \oplus v_5 \oplus v_3$	0 0 0	0 0 0
$v_3 \oplus v_6 \oplus v_1$	0 0 0	0 0 0
$v_4 \oplus v_5 \oplus v_6$	0 0 1	0 0 0

Table 1: The output of the players is depicted according to the constraint and variable sent respectively to Alice and Bob

This strategy gives the classical winning probability of 11/12. It is seen that the classical game value for this BCS is also 11/12 $\approx$ 0.916. This is the case because out of 12 possibilities, they will always answer at least one question incorrectly.

4.6 Quantum strategy

Alice and Bob decide to share an maximally entangled state $|\psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$. They decide that for inputs v_2 and v_6, their corresponding output would be 0, and they would give same output for v_1 and v_5, i.e $v_1 = v_5$. Now their set of constraints look like:

$$v_1 \oplus v_4 = 0 \tag{8}$$

$$v_1 \oplus v_3 = 0 \tag{9}$$

$$v_3 \oplus v_1 = 0 \tag{10}$$

$$v_1 \oplus v_4 = 1 \tag{11}$$

Thus it can be seen that Equations 8 $\&$ 11 constitute the CHSH inequality that was discussed previously. Equation 9 $\&$ 10 are the same.

1. Depending upon the constraint given, Alice either plays with the CHSH operators on her part of the shared state, or with the Equality operators, or if her constraint has either v_2 or v_6, she outputs the corresponding variable with 0 and plays either CHSH or Equality with the remaining variables in the constraint.

Constraint	Input	Alice's strategy	Bob's strategy
$v_1 \oplus v_4 \oplus v_2$	v_1 v_4 v_2	CHSH	CHSH CHSH 0
$v_2 \oplus v_5 \oplus v_3$	v_2 v_5 v_3	Equality	0 CHSH Equality
$v_3 \oplus v_6 \oplus v_1$	v_3 v_6 v_1	Equality	Equality 0 CHSH
$v_4 \oplus v_5 \oplus v_6$	v_4 v_5 v_6	CHSH	CHSH CHSH 0

Table 2: The table showing a complete quantum strategy for players Alice and Bob.

2. Bob also does the same. If his input is either v_2 or v_6, he outputs 0, for v_1, v_4 and v_5, he plays the CHSH game and plays with the Equality operator if he receives v_3.

3. The CHSH operators have been described in the previous XOR section and we know that the probablity of winning the CHSH game is 85%. Thus the players would succeed in scenarios 1st, 2nd, 11th and 12th in the table with 85% probability.

4. The Equality operator is $E = P_0 - P_1 = |0\rangle\langle 0| - |1\rangle\langle 1|$. Now from the table above, it is clear that there are two out of 12 cases when the players would apply the E operator. The probability that they would succeed is $\langle\psi| E \otimes E |\psi\rangle = 1$.

5. Thus from the table above, it is clear that in the 6 out 12 scenarios (namely scenarios 3rd, 4th, 6th, 7th, 8th and 12th), the players succeed with a probability 1.

6. In the two remaining scenarios, Alice operates with the E operator on her part of the shared state, while Bob operates with the corresponding CHSH operator. Two conditions may arise here with equal probability, either Bob gets the input 0 or 1.

7. The probability that Alice and Bob give the same output is given by $\frac{1}{2}(\langle\psi| P_0 \otimes B_0^0 |\psi\rangle + \langle\psi| P_1 \otimes B_1^0 |\psi\rangle + \langle\psi| P_0 \otimes B_0^1 |\psi\rangle + \langle\psi| P_1 \otimes B_1^1 |\psi\rangle) = \cos^2 \pi/8 \approx 0.85$.

8. Thus, this quantum strategy yeilds a value of 6/12 + 0.856/12 = 0.925.

Thus we see that this quantum strategy is provides a better winning probability than the best classical strategy.

5 Conclusion

The emphasis of this work was to understand non-local games with a class bigger than XOR games. We looked into one special class i.e. BCS games and tried to figure out the conditions under which it would admit a perfect strategy. For XOR games, the quantum game value can be efficiently computed using the semi-definite programming approach, but there is no such technique yet for BCS games. Hence instead of looking for quantum game value, we instead looked for a perfect strategy. We succeeded in specifying some conditions under which a perfect strategy cannot hold. We also looked at a particular example of BCS game where a quantum strategy gives a better value than the classical game value.

A related work could be to figure out an upper bound on the winning probability using a shared quantum state for BCS games. Navascues, Pirinio and Acin [6], talk about a hierarchy of semi-definite programs to test whether a correlation is quantum, and use this idea to give an upper bound on the quantum violation of Bell's Inequalities. This idea could be further developed to talk about the quantum violation of general non-local games as well.

References

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