Quantum Bayesian Games

Introduction

We are talking about a non-cooperative Bayesian game. The aim is that both the players have same average payoffs. Here, Classical and Quantum strategies would be developed for the bayesian game, and we would see that the Quantum approach necessarily gives an advantage over Classical approach.
The first half would focus on the game when the inputs by the referee is unbiased (i.e. each player has the same probability of receiving any input from the input set of the referee). In the second half, the referee biases his inputs and we look at different classical and quantum strategies.

Classical Approach

There are two players Alice(**A**) and Bob(**B**) who are playing the modified version of game of BoS(Battle of Sexes or, Bach or Stravansky). There is a referee who gives both the players a single bit(**s** or **t**), and they also have to give a single bit output (**a** or **b**) . These players share a coin(**r**) amongst themselves which can also take values 0 or 1.

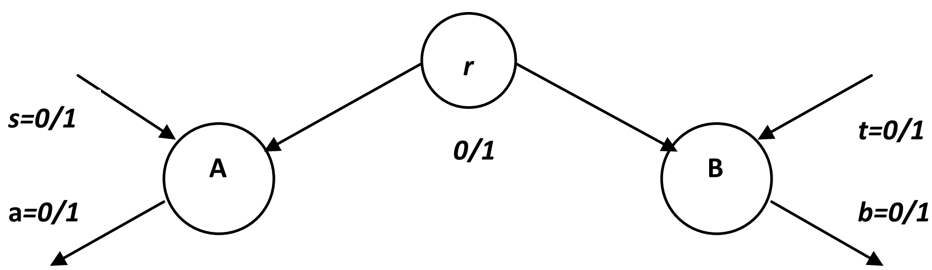


Fig. : This shows the entire model of the classical Bayesian game.

1. Keywords

Input: **s , t** $\in$ {0,1}

Output: **a , b** $\in$ {0,1}

Correlation: Coin **r** $\in$ {0,1}

Payoff : Alice: $\mathbf{P_1}$, Bob: $\mathbf{P_2}$

2. Game Explained

Now it has been informed to both the players that their payoff for the game is decided by the following matrices:

The table below shows the corresponding matrix that is used by the referee to compute the payoffs of Alice and Bob. The players also know the matrix components.

<table>
<tr><th>Input</th><th>Matrix Used</th></tr>
<tr><td>s=0 or t = 0 i.e. Input set {00, 01 ,10}.</td><td>MM_1: <table><tr><td>1, ½</td><td>0,0</td></tr><tr><td>0,0</td><td>½ ,1</td></tr></table></td></tr>
<tr><td>s=1 and t=1 i.e. Input set {11}.</td><td>M_2: <table><tr><td>0, 0</td><td>¾ , ¾</td></tr><tr><td>¾ , ¾</td><td>0,0</td></tr></table></td></tr>
</table>

In the matrices, the first entry denotes the payoff of player A, and second denotes that of player B. The matrix is understood as follows :

M_1		M_2	
a=0, b=0	$P_1 = 1, P_2 = ½$	a=0, b=0	$P_1 = 0, P_2 = 0$
a=1, b=0	$P_1 = 0, P_2 = 0$	a=1, b=0	P_1= ¾ , P_2= ¾
a=0, b=1	$P_1 = 0, P_2 = 0$	a=0, b=1	P_1 = ¾ , P_2 = ¾
a=1, b=1	P_1 = ½ , P_2 = 1	a=1, b=1	$P_1 = 0, P_2 = 0$

Classical Strategy

The aim of both the players is to maximize their average payoffs. So accordingly they follow the **Strategy** where Alice ouputs the input value(**s**) and Bob outputs the input complement(**t'**). :

Alice	Bob
a = s	**b = t' ***

* t' is the complement of t.

Next we look at the average payoff of both the players with this strategy:

s	t	a	b	Martix in operation	P_1	P_2
0	0	0	1	M1	0	0
0	1	0	0	M1	1	½
1	0	1	1	M1	½	1
1	1	1	0	M2	¾	¾
				Average Payoff:	1/4*(0+1+1/2+3/4) = 9/16	1/4*(0+1/2+1+3/4) = 9/16

Now we have to show that this strategy is indeed a Nash Equilibria:

In game theory, the **Nash equilibrium** is a solution concept of a non-cooperative game involving two or more players, in which each player is assumed to know the equilibrium strategies of the other players, and no player has anything to gain by changing only their own strategy. If each player has chosen a strategy and no player can benefit by changing strategies while the other players keep theirs unchanged, then the current set of strategy choices and the corresponding payoffs constitute a Nash equilibrium.

First, Alice's output(**a**) is fixed and it is checked if Bob's output (**b**) does indeed correspond to the maximum payoff if he sticks with the strategy:
The following cases may arrive:

a. If for Bob, **t**=0, then he is convinced that his payoff would be calculated w.r.t. matrix M_1.
b. Now, if Bob gets **t**=1, then he is no longer sure that w.r.t to which matrix payoff is calculated. He thinks that with probability of 0.5, Alice may get a 0(thus operation in M_1 matrix, and with probability of 0.5, Alice may get 1(thus operation in M_2 matrix). So he would check for his average payoff P_{2avg} , and if P_{2avg} increases by changing the output, then he would be tempted to change his strategy.

The following cases may arrive:

s	t	a=s (fixed)	Matrix in operation (if strategy is followed)	P_2 ,if b = t'	P_2 ,if b = t
0	0	0	M1(2nd column)	0.5*(0+1)	0.5*(1/2+0)
0	1	0	M1(0,0) or M_2(1,0)	0.5*(1/2+3/4)	0.5*(0+0)
1	0	1	M1(2nd column)	0.5*(0+1)	0.5*(1/2+0)
1	1	1	M1(0,0) or M_2(1,0)	0.5*(1/2+3/4)	0.5*(0+0)

So, we see that $\mathbf{P_2}$(b = t') > $\mathbf{P_2}$(b = t). This proves one half of the Nash Equilibrium.
For the other half, we fix Bob's output (b) and look is Alice's output (a) is indeed in Nash Equilibrium.

s	t	b=t' (fixed)	Matrix in operation (if strategy is followed)	P_2 ,if b = t'	P_2 ,if b = t
0	0	1	M1(1st Row)	0.5*(1+0)	0.5*(0+1/2)
0	1	0	M1(1st Row)	0.5*(1+0)	0.5*(0+1/2)
1	0	1	M1(1,1) or M_2(1,0)	0.5*(1/2+3/4)	0.5*(0+0)
1	1	0	M1(1,1) or M_2(1,0)	0.5*(1/2+3/4)	0.5*(0+0)

So we see that $\mathbf{P_1}$(a = s) > $\mathbf{P_1}$(a = s'). This proves that this strategy is indeed a classical *Nash Equilibrium*.

Quantum Strategy

With the Quantum Strategy, this game can be related to CHSH protocol. We know that for CHSH protocol, if both players share an entangled state $1/\sqrt{2}(|00\rangle + |11\rangle)$, then the probability of winning the game $a \oplus b = s \wedge t$, is 0.85. This means that if both the players Alice and Bob play the game with CHSH protocol, then:

s	t	Output (ab) with probability of 0.85	Matrix in operation	Payoffs (Alice , Bob)
0	0	00 or 11	M_1 (principle diagonal)	P_1= 0.85*1/2(1+ 1/2)= 0.85*3/4 P_1= 0.85*1/2(1/2+ 1)= 0.85*3/4
0	1	00 or 11	M_1 (principle diagonal)	P_1= 0.85*1/2(1+ 1/2)= 0.85*3/4 P_1= 0.85*1/2(1/2+ 1)= 0.85*3/4
1	0	00 or 11	M_1 (principle diagonal)	P_1= 0.85*1/2(1+ 1/2)= 0.85*3/4 P_1= 0.85*1/2(1/2+ 1)= 0.85*3/4
1	1	01 or 10	M_2 (off diagonal)	P_1= 0.85*1/2(3/4+ 3/4)= 0.85*3/4 P_1= 0.85*1/2(3/4+ 3/4)= 0.85*3/4
				P_{1avg}: ¼*(4*0.85*3/4) = 0.85*3/4 P_{2avg}: ¼*(4*0.85*3/4) = 0.85*3/4

So**,** $\mathbf{P_{1avg}}$(Quantum) = 0.85*3/4, where as $\mathbf{P_{1avg}}$(classical) = 9/16 = 0.75*3/4. Similarly for Bob. So Quantum technique does have an advantage over classical strategy.

Defining **P(ab|st),** as the probability of achieving the output ab, given the input st.
In the above calculation, it is assumed that P(00|00) and P(11|00) are the same and equals 0.85/2, and similarly for all different cases. But in reality, this may not be true.

In general, the Average payoffs for both the players would look like this:

st	P_1	P_2
00	P(00|00)*1 + P(11|00)*1/2	P(00|00)*1/2 + P(11|00)*1
01	P(00|01)*1 + P(11|01)*1/2	P(00|01)*1/2 + P(11|01)*1
10	P(00|10)*1 + P(11|10)*1/2	P(00|10)*1/2 + P(11|10)*1
11	P(01|11)*1/2 + P(10|11)*1	P(01|11)*1/2 + P(10|11)*1

For average payoff of individual players, we take the average of four values.

So experimentally we need to determine these values P(ab|st), and check if it really is better than classical strategy every time.

Experimental Technique:

The entanglement source was set in order to get the number of counts to be approximately the same when both the polarizers are setup in 45 degrees and 315 degrees. This ensured the near perfect alignment of the setup.

The goal was just to perform the CHSH experiment and extract the individual probabilities from there. For the CHSH, the angles (basis) chosen were the following:

Alice			Bob		
s = 0	Z+	0 deg.	t = 0	A+	337.5 deg
	Z-	90 deg.		A-	67.5 deg
s = 1	X+	315 deg	t = 1	B+	22.5 deg
	X-	45 deg.		B-	112.5 deg

Now for a given **s** , if Alice measures with **+** angle, then this implies her output **a** = 0, and similarly for **-** angle , **a** = 1. Similarly for Bob.

This way, we could get the data for all combinations of {st} and the coincidence counts for all possible output combinations for Alice and Bob.

Calculating Probabilities:

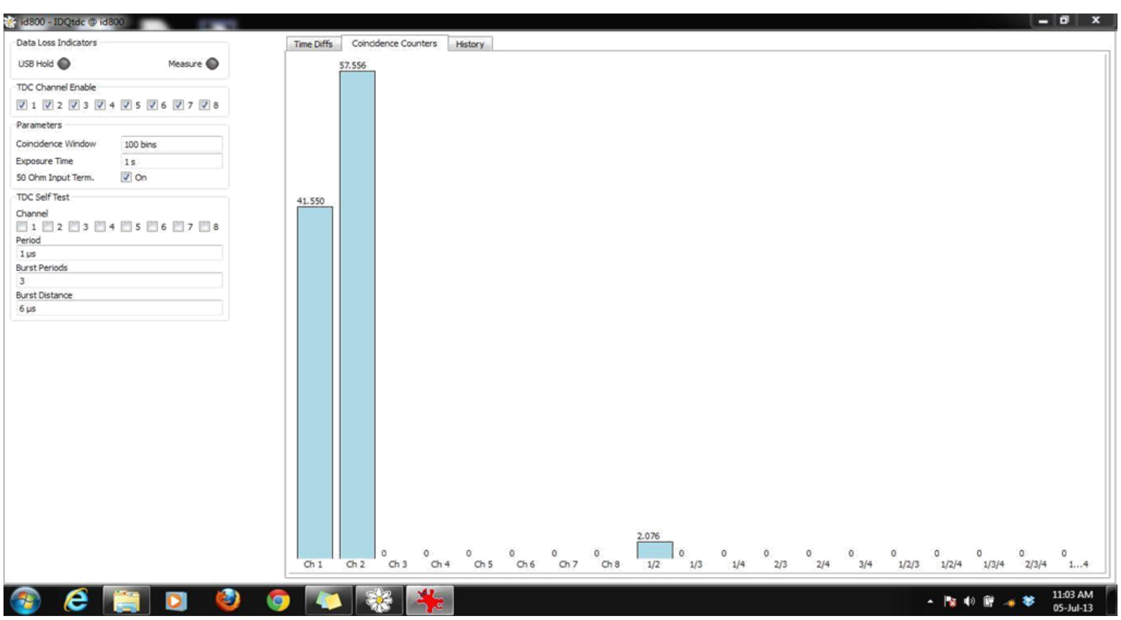
Let's say we need to calculate the P(00|00), i.e for input *st*: 00 , which is the probability of getting the output *ab* : 00. Alice measures in 'Z' basis and Bob in 'A' basis. So we get the coincidence count for Z+A+, Z+A-, Z-A- and Z-A+ settings. Now,

$$P(00|00) = \text{Count}(Z+A+)/(\text{Count}(Z+A+)+\text{Count}(Z+A-)+\text{Count}(Z-A-)+\text{Count}(Z-A+))$$

Case 1: Without removing the stray counts.

In this case, we just take the outputs from the two detectors and read the coincidence counts. The output of the detectors is given to the TDC(Time to digital convertor), and it displays the real time histogram showing the individual counts and the coincidence counts. It looks something like this:

Fig: A snapshot of case where *st :* 00 and Alice and Bob are measuring in Z+ and A+ respectively. i.e *ab :* 00



With the TDC, the exposure time was kept at 1 sec , and the coincidence window was set at 100 Bins = 8.1 ns. (1 Bin = 81 ps)

A software *IrfanView* (http://www.irfanview.com/)was used to take 100 snapshots of the histogram per second. This technique was employed for all different combinations of *st* and *ab.*

Case2: Removing the stray counts

Until now, the coincidences from two detectors were used to calculate the respective probabilities. But this may as well have contributions from dark counts. This could be removed by introducing a certain delay along one of the channels. Since the coincidence window is 8.1 ns. So the minimum length of wire needed to introduce the delay is 2.42 m. (8.1ns x speed of light). (*I do not remember the exact length of wire introduced while performing the experiment, but i am pretty sure it is more that 2.5 m*)

So the delay was introduced along the first detector and the output was taken on channel 3 of TDC. A snapshot of it looked like this:

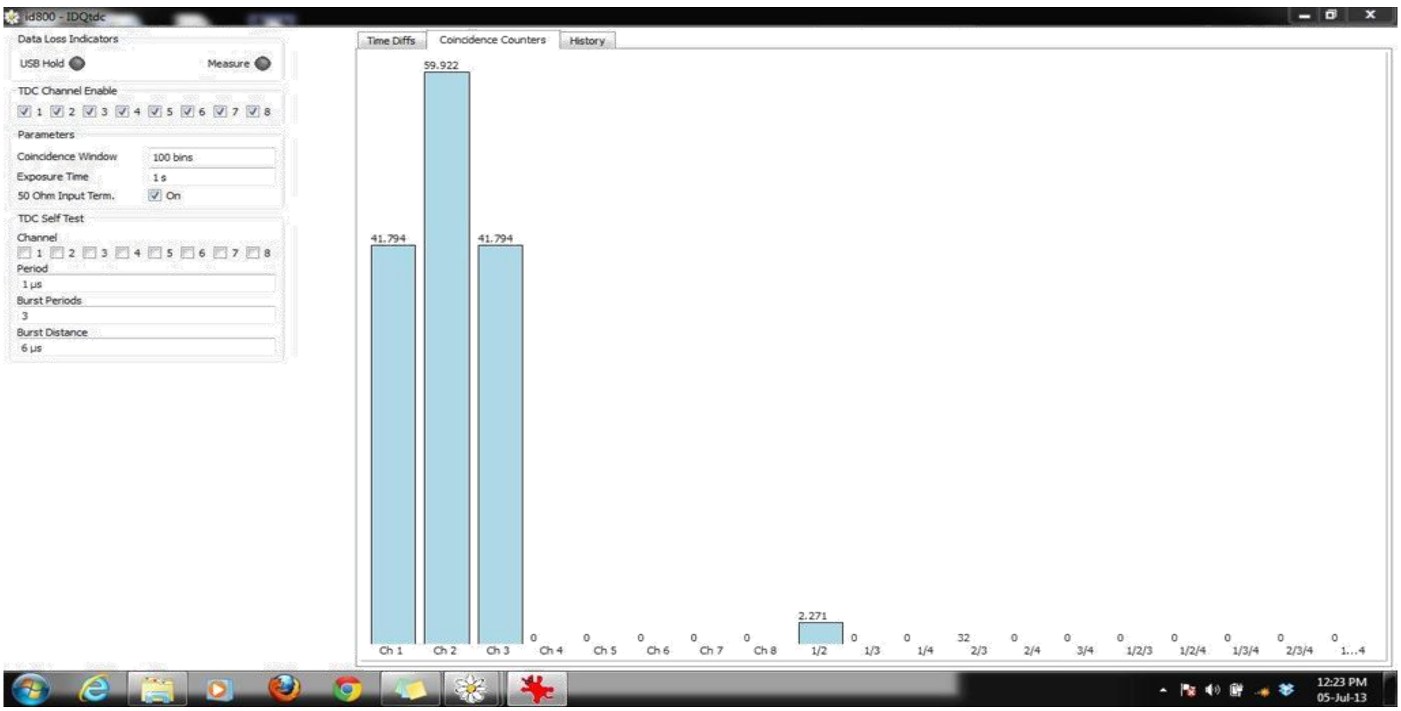


Fig: A snapshot of case where *st :* 00 and Alice and Bob are measuring in Z+ and A+ respectively. i.e *ab :* 00

Ch1: Normal output from detector 1.

Ch2 : Normal output from detector2.

Ch3 : Delayed output from detector 1.

1/2 : Coincidence between Ch1 and Ch2

2/3 : Coincidence between Ch3 and Ch2. (Stray count)

So while calculating the probabilities, the stray count is subtracted from the 1/2 count, to get actual coincidence count.

Biased Quantum Bayesian Game

Classical Approach

Earlier, the referee gave inputs with the probability of Pr(0|s) = 0.5 and Pr(0|t) = 0.5. That was the unbiased case. But now we would look at the situation when the referee starts to bias his inputs. Here Pr(0|s) is the probability of referee giving input 0 to Alice, and similarly Pr(0|t) is the probability of referee giving input 0 to Bob.

Let's say, **Pr(0|s) = p** and **Pr(1|s) = 1-p** ; **Pr(0|t) = q** and **Pr(1|t) = 1-q**.
This implies:
Pr(00|st) **= pq** , **Pr**(01|st) **= p(1-q)** , **Pr**(10|st) **= (1-p)q** , **Pr**(11|st) **= (1-p)(1-q)** .
Here Pr(00|st) is the probability of referee giving input 0 to Alice and 0 to Bob.

The payoffs for Alice and Bob are determined wrt the matrices:

<table>
<tr><th>Input</th><th>Matrix Used</th></tr>
<tr><td>s=0 or t = 0 i.e. Input set {00, 01 ,10}.</td><td>M_1: <table><tr><td>1, ½</td><td>0,0</td></tr><tr><td>0,0</td><td>½ ,1</td></tr></table></td></tr>
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Classical Strategy

Different strategies were checked for the Nash Equlibrium(NE) condition:
(a,b) ≡ {(0,0) , (0,1) , (1,0) , (1,1) , (0,r) , (1, r) , (r,0) , (r,1) , (r,r) , (s,t) , (s,t') , (s',t) } ,
r is a random coin that players share and can take values {0,1}. **t'** is the complement of **t** and **s'** is **s** complement. Among them, the strategies (0,0), (1,1), (r,r) , (s,t') ,(s',t) are in a NE for certain **p** and **q** ranges.

Some explanation as why other strategies are not in NE:

(a, b)	Problematic Cases
(0,1)	For input (s,t) ≡ (0,0), keeping **a** fixed, Bob knows that his payoff is calculated from M1. So he would change **b** from 1 to 0 to gain more. Not in NE
(1,0)	For input (s,t) ≡ (0,0), keeping **a** fixed, Bob knows that his payoff is calculated from M1. So he would change **b** from 0 to 1 to gain more. Not in NE
(0,r)	For input (s,t) ≡ (0,0), keeping **a** fixed, if **r**=1, Bob knows that his payoff is calculated from M1. So he would change **b** from 1 to 0 to gain more. Not in NE

(1,r)	For input (s,t) ≡ (0,0), keeping **a** fixed, if **r**=0, Bob knows that his payoff is calculated from M1. So he would change **b** from 0 to 1 to gain more. Not in NE
(r,0)	For input (s,t) ≡ (0,0), keeping **b** fixed, if **r**=1, Alice knows that her payoff is calculated from M1. So she would change **a** from 1 to 0 to gain more. Not in NE
(r,1)	For input (s,t) ≡ (0,0), keeping **b** fixed, if **r**=0, Alice knows that her payoff is calculated from M1. So she would change **a** from 0 to 1 to gain more. Not in NE
(s,t)	For input (s,t) ≡ (0,1), keeping **a** = s fixed, Bob thinks that with **p** probability, his payoff is calculated wrt to M1 and with **1-p**, his payoff is calculated wrt M2. P_2(t)= p*0+(1-p)*0 , whereas P_2(t')= p*1/2+(1-p)*3/4. So he changes his strategy.

<u>Some Conditional Nash Equilibrium Strategies</u>:

- **<u>(a,b) ≡ (0,0)</u>** :

First we keep **a** fixed and look at Bob's expected payoff with his **b** strategy and **b'** strategy.
Similarly, next **b** is kept fixed and Alice's expected payoff is looked with **a** and **a'** strategy:

s	t	a(fixed)	b	P_B(b)	P_B(b')	Remark
0	0	0	0	1/2	0	Bob knows his P_B is calculated wrt M1
0	1	0	0	p*1/2+(1-p)*0	p*0+(1-p)*3/4	Bob knows with prob. p, he is in M1 and (1-p) in M2
1	0	0	0	1/2	0	Bob knows his P_B is calculated wrt M1
1	1	0	0	p*1/2+(1-p)*0	p*0+(1-p)*3/4	Bob knows with prob. p, he is in M1 and (1-p) in M2

NE Condition: P_B(b) ≥ P_B(b') ≡ p≥3/5

s	t	a	b(fixed)	P_A(a)	P_A(a')	Remark
0	0	0	0	1	0	Alice knows her P_A is calculated wrt M1
0	1	0	0	1	0	Alice knows her P_A is calculated wrt M1
1	0	0	0	q*1+(1-q)*0	q*0+(1-q)*3/4	Alice know with prob. q, she is in M1 and (1-p) in M2
1	1	0	0	q*1+(1-q)*0	q*0+(1-q)*3/4	Alice know with prob. q, she is in M1 and (1-p) in M2

NE Condition: P_A(a) ≥ P_A(a') ≡ q ≥ 3/7

The expected payoffs of players with this strategy are:
P_A = pq*1 + p(1-q)*1 + (1-p)q*1 + (1-p)(1-q)*0 = **p+q-pq**
P_B = pq*1/2 + p(1-q)*1/2 + (1-p)q*1/2 + (1-p)(1-q)*0 = **½*(p+q-pq)**

- **<u>(a,b) ≡ (1,1)</u>** :

s	t	a(fixed)	b	P_B(b)	P_B(b')	Remark
0	0	1	1	1	0	Bob knows his P_B is calculated wrt M1
0	1	1	1	p*1+(1-p)*0	p*0+(1-p)*3/4	Bob knows with prob. p, he is in M1 and (1-p) in M2
1	0	1	1	1	0	Bob knows his P_B is calculated wrt M1
1	1	1	1	p*1+(1-p)*0	p*0+(1-p)*3/4	Bob knows with prob. p, he is in M1 and (1-p) in M2

NE Condition: P_B(b) ≥ P_B(b') ≡ p≥3/7

s	t	a	b(fixed)	$P_A(a)$	$P_A(a')$	Remark
0	0	1	1	½	0	Alice knows her P_A is calculated wrt M1
0	1	1	1	½	0	Alice knows her P_A is calculated wrt M1
1	0	1	1	q*1/2 +(1-q)*0	q*0+(1-q)*3/4	Alice know with prob. q, she is in M1 and (1-p) in M2
1	1	1	1	q*1/2+(1-q)*0	q*0+(1-q)*3/4	Alice know with prob. q, she is in M1 and (1-p) in M2

NE Condition: $P_A(a) \geq P_A(a') \equiv q \geq 3/5$

$\mathbf{P_A}$ = pq*1/2 + p(1-q)*1/2 + (1-p)q*1/2 + (1-p)(1-q)*0 = **½*(p+q-pq)**
$\mathbf{P_B}$ = pq*1 + p(1-q)*1 + (1-p)q*1 + (1-p)(1-q)*0 = **p+q-pq**

- **<u>(a,b) ≡ (r,r)</u>** :

s	t	a(fix)=r	b=r	$P_B(b)$	$P_B(b')$	Remark
0	0	0	0	½	0	Bob knows his P_B is calculated wrt M1
0	0	1	1	1	0	Bob knows his P_B is calculated wrt M1
0	1	0	0	p*1/2+(1-p)*0	p*0+(1-p)*3/4	Bob knows with prob. p, he is in M1 and (1-p) in M2
0	1	1	1	p*1+(1-p)*0	p*0+(1-p)*3/4	Bob knows with prob. p, he is in M1 and (1-p) in M2
1	0	0	0	½	0	Bob knows his P_B is calculated wrt M1
1	0	1	1	1	0	Bob knows his P_B is calculated wrt M1
1	1	0	0	p*1/2+(1-p)*0	p*0+(1-p)*3/4	Bob knows with prob. p, he is in M1 and (1-p) in M2
1	1	1	1	p*1+(1-p)*0	p*0+(1-p)*3/4	Bob knows with prob. p, he is in M1 and (1-p) in M2

NE Condition: $P_B(b) \geq P_B(b') \equiv p \geq 3/5$

s	t	a	b(fixed)	$P_A(a)$	$P_A(a')$	Remark
0	0	0	0	1	0	Alice knows her P_A is calculated wrt M1
0	0	1	1	1/2	0	Alice knows her P_A is calculated wrt M1
0	1	0	0	1	0	Alice knows her P_A is calculated wrt M1
0	1	1	1	1/2	0	Alice knows her P_A is calculated wrt M1
1	0	0	0	q*1+(1-q)*0	q*0+(1-q)*3/4	Alice know with prob. q, she is in M1 and (1-p) in M2
1	0	1	1	q*1/2+(1-q)*0	q*0+(1-q)*3/4	Alice know with prob. q, she is in M1 and (1-p) in M2
1	1	0	0	q*1+(1-q)*0	q*0+(1-q)*3/4	Alice know with prob. q, she is in M1 and (1-p) in M2
1	1	1	1	q*1/2+(1-q)*0	q*0+(1-q)*3/4	Alice know with prob. q, she is in M1 and (1-p) in M2

NE Condition: $P_A(a) \geq P_A(a') \equiv q \geq 3/5$

$\mathbf{P_A}$ = pq*3/4 + p(1-q)*3/4 + (1-p)q*3/4 + (1-p)(1-q)*0 = **3/4*(p+q-pq)**
$\mathbf{P_B}$ = pq*3/4 + p(1-q)*3/4 + (1-p)q*3/4 + (1-p)(1-q)*0 = **3/4*(p+q-pq)**

- **(a,b) ≡ (s,t')** :

s	t	a(fixed)	b	$P_B(b)$	$P_B(b')$	Remark
0	0	0	1	p*0+(1-p)*1	p*1/2+(1-p)*0	Bob knows his P_B is calculated wrt M1 and with prob p, s=0, and with (1-p), s=1
0	1	0	0	p*1/2+(1-p)*3/4	p*0+(1-p)*0	Bob knows with prob. p, he is in M1 and (1-p) in M2
1	0	1	1	p*0+(1-p)*1	p*1/2+(1-p)*0	Bob knows his P_B is calculated wrt M1 and with prob p, s=0, and with (1-p), s=1
1	1	1	0	p*1/2+(1-p)*3/4	p*0+(1-p)*0	Bob knows with prob. p, he is in M1 and (1-p) in M2

NE Condition: $P_B(b) \geq P_B(b') \equiv p \leq 2/3$

s	t	a	b(fixed)	$P_A(a)$	$P_A(a')$	Remark
0	0	0	1	q*0+(1-q)*1	q*1/2+(1-q)*0	Alice knows her P_A is calculated wrt M1 and with prob q, b=1, and with (1-p), b=0
0	1	0	0	q*0+(1-q)*1	q*1/2+(1-q)*0	Alice knows her P_A is calculated wrt M1 and with prob q, b=1, and with (1-p), b=0
1	0	1	1	q*1/2+(1-q)*3/4	q*0+(1-q)*0	Alice know with prob. q, she is in M1 and (1-q) in M2
1	1	1	0	q*1/2+(1-q)*3/4	q*0+(1-q)*0	Alice know with prob. q, she is in M1 and (1-q) in M2

NE Condition: $P_A(a) \geq P_A(a') \equiv q \leq 2/3$

$\mathbf{P_A}$ = pq*0 + p(1-q)*1 + (1-p)q*1/2 + (1-p)(1-q)*3/4 = **3/4 + p/4 – q/4 -3pq/4**
$\mathbf{P_B}$ = pq*0 + p(1-q)*1/2 + (1-p)q*1 + (1-p)(1-q)*3/4 = **3/4 + q/4 – p/4 - 3pq/4**

- **(a,b) ≡ (s',t)** :

s	t	a(fixed)	b	$P_B(b)$	$P_B(b')$	Remark
0	0	1	0	p*0+(1-p)*1/2	p*1+(1-p)*0	Bob knows his P_B is calculated wrt M1 and with prob p, a=1, and with (1-p), a=0.
0	1	1	1	p*1/2+(1-p)*3/4	p*0+(1-p)*0	Bob knows with prob. p, he is in M1 and (1-p) in M2
1	0	0	0	p*0+(1-p)*1/2	p*1+(1-p)*0	Bob knows his P_B is calculated wrt M1 and with prob p, a=1, and with (1-p), a=0.
1	1	0	1	p*1/2+(1-p)*3/4	p*0+(1-p)*0	Bob knows with prob. p, he is in M1 and (1-p) in M2

NE Condition: $P_B(b) \geq P_B(b') \equiv p \leq 1/3$

s	t	a	b(fixed)	$P_A(a)$	$P_A(a')$	Remark
0	0	1	0	q*0+(1-q)*1/2	q*1+(1-q)*0	Alice knows her P_A is calculated wrt M1 and with prob q, b=1, and with (1-p), b=0
0	1	1	1	q*0+(1-q)*1/2	q*1+(1-q)*0	Alice knows her P_A is calculated wrt M1 and with prob q, b=1, and with (1-p), b=0
1	0	0	0	q*1+(1-q)*3/4	q*0+(1-q)*0	Alice know with prob. q, she is in M1 and (1-p) in M2
1	1	0	1	q*1+(1-q)*3/4	q*0+(1-q)*0	Alice know with prob. q, she is in M1 and (1-p) in M2

NE Condition: $P_B(b) \geq P_B(b') \equiv q \leq 1/3$

P_A = pq*0 + p(1-q)*1/2 + (1-p)q*1 + (1-p)(1-q)*3/4 = **3/4 + q/4 – p/4 - 3 pq/4.**
P_B = pq*0 + p(1-q)*1 + (1-p)q*1/2 + (1-p)(1-q)*3/4 = **3/4 + p/4 – q/4 – 3pq/4.**

Summary of all Classical NE strategies:

Strategy	NE Condition	P_A	P_B
(a,b) ≡ (0,0)	p≥3/5 , q≥3/7	p+q-pq	1/2*(p+q-pq)
(a,b) ≡ (1,1)	p≥3/7 , q≥3/5	1/2*(p+q-pq)	p+q-pq
(a,b) ≡ (r,r)	p≥3/5 , q≥3/5	3/4*(p+q-pq)	3/4*(p+q-pq)
(a,b) ≡ (s,t')	p≤2/3 , q≤2/3	3/4 + p/4 – q/4 -3pq/4	3/4 + q/4 – p/4 - 3 pq/4
(a,b) ≡ (s',t)	p≤1/3 , q≤1/3	3/4 + q/4 – p/4 - 3 pq/4	3/4 + p/4 – q/4 -3pq/4

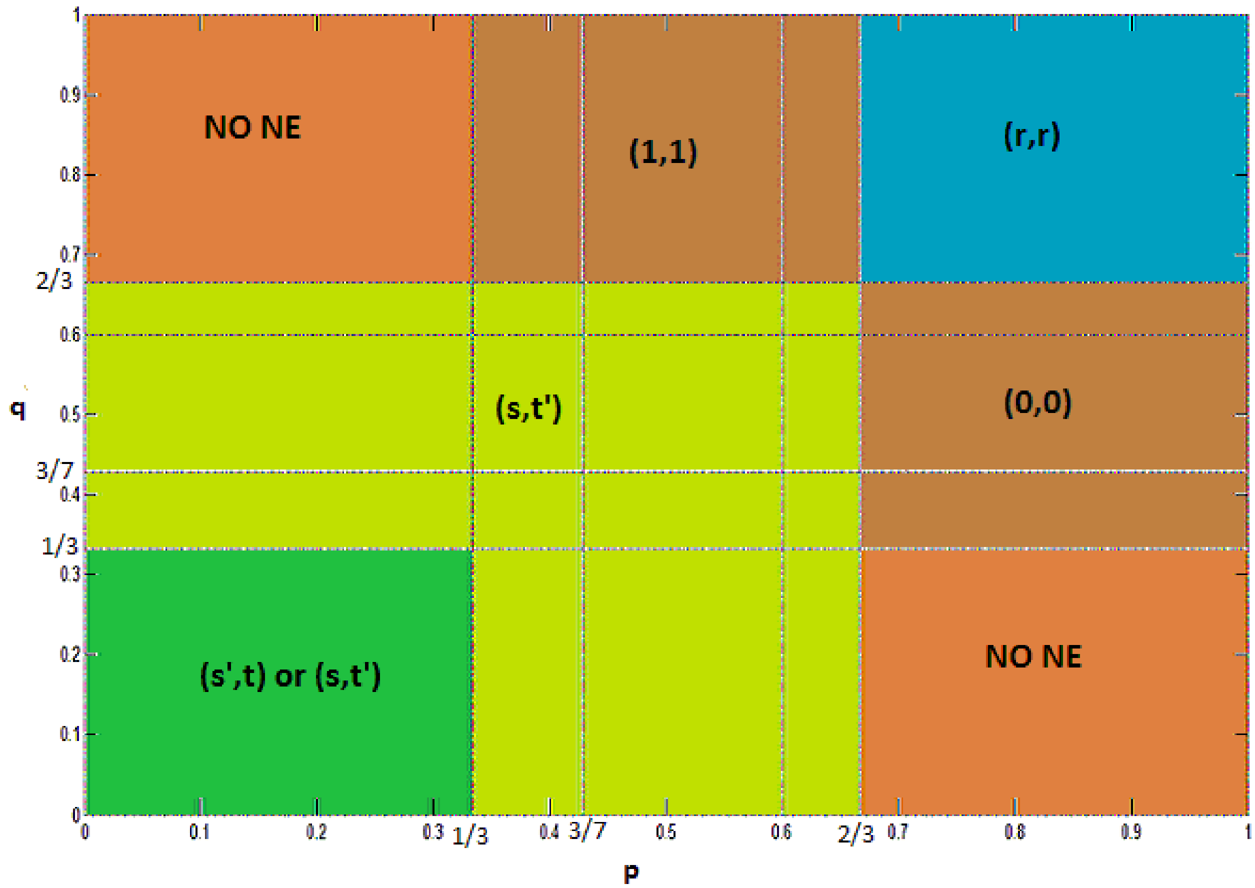

Figure 2: Figure depicting optimal strategies for different p and q values

The green shaded region does not have a clear optimal strategy because the payoff for player A with one strategy is equal to payoff for player B with the other strategy and vice versa. So while one gains more if both stick with one strategy, the other is better off if they both would agree on the other strategy. Both are in NE in the green region, and they have to decide beforehand (by a coin toss or something else) as to which strategy to stick to.

The two orange regions does not have any stratrgy in Nash Equilibrium.

Quantum Strategy

In this case, Alice and Bob share an entanglement pair, $|\Psi\rangle = 1/\sqrt{2}\,(|00\rangle + |11\rangle)$, and they have an the observable A_i , B_i ,for $i \in \{0,1\}$, for inputs $s,t \in \{0,1\}$ respectively. Here we are considering the biased case i.e. **Pr(0|s) = p** and **Pr(1|s) = 1-p** ; **Pr(0|t) = q** and **Pr(1|t) = 1-q**.

The expression to be maximized is:

CHSH[p,q] = $pqE(A_0,B_0) + p(1-q)E(A_0,B_1) + (1-p)qE(A_1,B_0) - (1-p)(1-q)\ (A_1,B_1)$, where the E's correspond to joint strategy corresponding to a given *st*.

For the quantum case:

$E1 = E(A0,B0) = \langle\Psi|A0\otimes B0|\Psi\rangle$
$E2 = E(A0,B1) = \langle\Psi|A0\otimes B1|\Psi\rangle$,
$E3 = E(A1,B0) = \langle\Psi|A1\otimes B0|\Psi\rangle$,
$E4 = E(A1,B1) = \langle\Psi|A1\otimes B1|\Psi\rangle$

Building upon the definition of the paper [1], we see that:

Conditions	CHSH[p,q]
$p\geq 1/2$, $q\geq 1/2$ and $pq \geq 1/2$	$\leq -1 + 2(p + q - pq)$ (Classical Bound)
$p\geq 1/2$, $q\leq 1/2$ and $p(1-q) \geq 1/2$	
$p\leq 1/2$, $q\leq 1/2$ and $qp \geq 1/2$	
$p\leq 1/2$, $q\geq 1/2$ and $q(1-p) \geq 1/2$	
$p\geq 1/2$, $q\geq 1/2$ and $pq < 1/2$	$\leq \sqrt{2}\sqrt{(p^2 + (1-p)^2)}\sqrt{(q^2 + (1-q)^2)}$ (Quantum Bound)
$p\geq 1/2$, $q\leq 1/2$ and $p(1-q) < 1/2$	
$p\leq 1/2$, $q\leq 1/2$ and $qp < 1/2$	
$p\leq 1/2$, $q\geq 1/2$ and $q(1-p) < 1/2$	

So, the Quantum strategy is good only in certain areas shown above, because we know that in the region outside this, we have a classical strategy which always score better(or atleast as good) than the Quantum strategy.

The observables that achieve the CHSH_Q [p,q] bound are:

$$A_0 = \frac{X(q+(1-q)\cos\beta) + Z(1-q)\sin\beta}{\sqrt{(q+(1-q)cos\beta)^2+\left((1-q)sin\beta\right)^2}}$$

$$A_1 = \frac{X(q-(1-q)\cos\beta) - Z(1-q)\sin\beta}{\sqrt{(q-(1-q)cos\beta)^2+\left((1-q)sin\beta\right)^2}}$$

$B_0 = X$
$B_1 = X\cos\beta + Z\sin\beta$

where, X and Z are standard pauli matrices and,

$$cos\beta = \frac{1}{2}\ \frac{(q^2+(1-q)^2)(p^2+(1-p)^2)}{q(1-q)(p^2+(1-p)^2)}$$

Given E_1, E_2, E_3 and E_4, the Pr(a⊕b =st|st) is the probability of getting correct output for the input st. Pr(a⊕b =0 |00) = 1/2 + E_1/2 .
Pr(a⊕b =0 |01) = 1/2 + E_2/2
Pr(a⊕b =0 |00) = 1/2 + E_3/2
Pr(a⊕b =1 |00) = 1/2 – E_4/2

So, for our set of matrices, the average Quantum payoff's for Alice and Bob with quantum strategy are:

P_A = pq*(1/2 + E1/2)*3/4 + p(1-q)* (1/2 + E2/2)*3/4 + (1-p)q*(1/2 + E3/2)*3/4 + pq*(1/2 – E4/2)*3/4

$$= \frac{3}{4}\left(\frac{1}{2} + \frac{\sqrt{(p^2 + (1-p)^2)(q^2 + (1-q)^2)}}{\sqrt{2}}\right)$$

P_B = pq*(1/2 + E1/2)*3/4 + p(1-q)* (1/2 + E2/2)*3/4 + (1-p)q*(1/2 + E3/2)*3/4 + pq*(1/2 – E4/2)*3/4

$$= \frac{3}{4}\left(\frac{1}{2} + \frac{\sqrt{(p^2 + (1-p)^2)(q^2 + (1-q)^2)}}{\sqrt{2}}\right)$$

References:

[1] : Biased nonlocal quantum games, http://arxiv.org/abs/1011.6245