Quantum Non-Local Games

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This work investigates upon the idea of using quantum entanglement in reference to non-local(NL) games. We begin with the definition of a non-local game and study it's importance in context to the advantage gained over playing the game classically i.e. when no entanglement is involved. For certain NL games, like the XOR games, the gap between the value of the game(maximum success probability) played classically and other using entanglement sharing,can be computed efficiently. But for a general class of NL games, while determining the classical value of the game is NP-Hard, the problem of computing the quantum value is not known to be decidable. Even deciding whether there is a perfect quantum strategy is not known to be decidable. We consider a special class of NL games, namely the Binary Constraint System(BCS) games, and look whether they admit a perfect strategy or not. Next, we take a BCS game and give a quantum strategy which has a success probability higher than that of the classical value of that game.

I. INTRODUCTION

A non-local game consists of two *players* (Alice & Bob) who cooperate with each other to jointly win the game. A *referee* runs the game and all the communication during the game is between the referee and players- no inter player communication is allowed once the game starts. The referee has his distribution of *questions*, and randomly selects one question for each players and sends each question to the appropriate player. Now, the players send back *answers* to the refree and based on the answers and questions, the referee decides whether the players win or lose the game. The refree's distribution of the questions and the winning predicate is publicly known to the players.

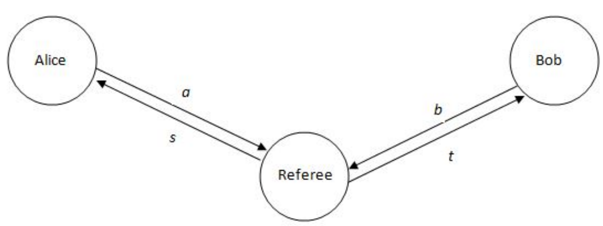

FIG. 1. Figure illustrating a non-local game with two players

A. Game Definition

An NL game is defined as G = G(π,V), where π is the probablity distribution on the question set (S×T) of referee and the V is the predicate on (S×T×A×B), such that V is 1 when both the players answer the question correctly and 0 otherwise. The referee chooses the questions (s, t) $\in$ (S×T) randomly from the distribution π, and sends s to Alice and t to Bob. Alice then replies with $a \in$ A and similarly Bob replies with $b \in$ B. Note that upon receiving the questions, the players are not allowed to communicate with each other. The predicate V on (s,t,a,b) is expressed as V(a,b|s,t) to stress upon the fact whether (a,b) is correct or not given (s,t).

B. Game Value

A game value is defined as the maximum success probability achievable by the players. Now there are two ways for the players to play the game:

1. Either they choose to share a *quantum* state, in which case the game value is denoted by ω_q.

2. Or, they do not share a quantum state and try to work out the best possible *classical* strategy beforehand, in which case the game value is denoted by ω_c.

The *classical* value of the game is defined as follows:

$$\omega_c(G) = \max_{a,b} \sum_{s,t} \pi(s,t) V(a, b \mid s, t) Pr(a, b \mid s, t)$$

The maximization is over all possible strategies including the probabilistic strategies. But, as a probablistic strategy value can be written as convex combination of the deterministic strategies, hence the maximum value of the game can be achieved using a deterministic strategy. Hence from here, $a \to a(s)$ and $b \to b(t)$.
A *quantum* strategy is defined by a shared entagled state $|\psi\rangle \in \mathbb{C}^{d\times d}$ for $d \geq 1$ between the players Alice and Bob who use projection operators to measure their part of the quantum state. The table below describes a quantum strategy for Alice and Bob:

TABLE I. Table showing the quantum strategies for 2 players

Alice	Bob
$\{A_a^s\}$, $\bigvee$ question s.	$\{B_b^t\}$, $\bigvee$ question t.
${A_a^s}^\dagger = A_a^s$.	${B_b^t}^\dagger = B_b^t$.
$\sum_a A_a^s = I$	$\sum_b B_b^t = I$

The probability that on questions *s,t* Alice answers a and Bob answers b is given by $\langle\psi|A^s_a \otimes B^t_b|\psi\rangle$. The *entangled* value of the game ω_q is given by:

$$\omega_q(G(V,\pi)) = \max_{|\psi\rangle,|A^s_a,B^t_b} \sum_{a,b,s,t} \pi(s,t)V(a,b \mid s,t)\langle\psi|A^s_a{\otimes}B^t_b|\psi\rangle$$

Note that instead of projection operators, we could have started with a much more generalized POVM operators, but any POVM operators can be expressed as a projection operator in a higher dimensional hilbert space and since here we are not restricting the dimensionality of the hilbert space, hence there is no loss of generality in assuming them to be projection operators.

II. BELL'S INEQUALITY

In 1935, Einstein, Podolsky and Rosen, developed a thought experiment, they called it the EPR Paradox where they imagined two initially interacting physical systems described by a single quantum state $|\psi\rangle$. Once separated, the two systems(lets say *photons*) are still described by $|\psi\rangle$, and a measurement of an observable in one system instantaneuosly affects the state of corresponding observable in the second system even though the systems are no longer physically linked and may be light years separated. Several hidden variable theories came up, which claimed to describe such a scenario.
In 1964, [10] Bell proposed a mechanism to look for the hidden variables using his famous Bell's inequality(BI). A violatoin of these inequalities would lead to failure of such a theory. He introduced entanglement which could perfectly explain the EPR paradox and would lead to the violation of Bell's ineqality, thus discarding the concept of hidden variables and introducing a new central feature in QM .
Non Local Games provide a natural setting for describing Bell's inequality. BI is analogous to the probablity of winning the NL game using the best possible classical strategy. Its *violation* would mean winning the NL game using a quantum strategy whose game value would be higher than the classical game value. Thus a study of gaps in NL games is of high relevance to quantum information.
Bell's inequality introduces *correlations* between outputs of two players. For a given set of inputs and outputs, classical correlations form a convex polytope (Figure 2) with fixed vertices corresponding to deterministic strategies. Quantum correlations do not have finite end points, hence these strategies are not known to be decidable. Here NL is a set which contains all non-local correlations, including the non-signalling.

III. CHSH XOR GAME

This is a non-local game where the quantum game value can be efficiently determined. This game consists

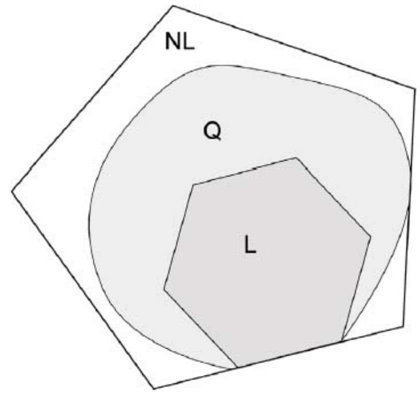

FIG. 2. [4] Figure showing the correlations with L being the classical region and admits a finite set of vertices(deterministic strategies), while the Q region has no finite vertices, thus showing that the problem of finding the best strategy is undecidable.NL is global set which contains all correlations.

of two players Alice and Bob. The referee's distribution π is uniform with $st \in \{00, 01, 10, 11\}$. Alice would have to respond with a single bit a and similarly Bob responds with b. The game winning condition is $a \oplus b = s \wedge t$.

st	a ⊕ b
00	0
01	0
10	0
11	1

For this game, $\omega_c = 3/4$, because they will always answer atleast one question incorrectly. One such strategy is when both the players output 0 in all the cases.
However, it will be shown that the quantum game value for this NL game is $\cos^2(\pi/8) \approx 0.85$.
Alice and Bob share a quantum state $|\psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$. The set of operators $\{A^s_a\}$ and $\{B^t_b\}$ for Alice and Bob are defined as follows[2] :

$$A^0_a = |\phi_a(0)\rangle\langle\phi_a(0)|,$$
$$A^1_a = |\phi_a(\pi/4)\rangle\langle\phi_a(\pi/4)|,$$
$$B^0_b = |\phi_b(\pi/8)\rangle\langle\phi_a(\pi/8)|,$$
$$B^0_b = |\phi_b(-\pi/8)\rangle\langle\phi_a(-\pi/8)|$$

where,

$$|\phi_0(\theta)\rangle = \cos(\theta)|0\rangle + \sin(\theta)|1\rangle,$$
$$|\phi_1(\theta)\rangle = -\sin(\theta)|0\rangle + \cos(\theta)|1\rangle$$

So, the upon receiving 0, Alice operates on her part of the shared state with $A^0 = A^0_0 - A^0_1$, and with $A^1 = A^1_0 - A^1_1$ on getting 1. Similarly, Bob has operators $B^0 = B^0_0 - B^0_1$ and $B^1 = B^1_0 - B^1_1$ for inputs 0 and1 respectively. Now since the probability that on questions *s,t* Alice answers a and Bob answers b is given by $\langle\psi|A^s_a \otimes B^t_b|\psi\rangle$, hence the maximum success probablity can be easlily computed for this game and it comes out to be $\approx 85\%$ [2].

IV. BINARY CONSTRAINT SYSTEM GAME

For the XOR game considered previously, the quantum game value is computed efficiently using the semi-definite programming tool[2] [6]. But for other classes of non-local games, determining the game value is undecidable. Even determining whether a perfect strategy exists is non known to be decidable as there is no general structure to look for such a strategy. So, instead of looking for a game value (which would be a tougher problem to deal with initially), we would look for some characterization under which a perfect strategy exists. A special class of NL games i.e. the BCS game is looked at in this section.

A. Game Definition

A *binary constraint system* (BCS) consists of n binary variables, $v_1, v_2, ..., v_n$, and m constraints, $c_1, c_2, ..., c_m$, where each c_j is a binary-valued function of a subset of the variables. We take an example of BCS with $n = 6$ and $m = 4$:

$$v_1 \oplus v_4 \oplus v_2 = 0$$
$$v_2 \oplus v_5 \oplus v_3 = 0$$
$$v_3 \oplus v_6 \oplus v_1 = 0$$
$$v_4 \oplus v_5 \oplus v_6 = 1$$

In the example shown above, all the constraints are only the fucntion of parity of variables. Hence, this special BCS is called the *parity* BCS. This above example will be studied in detail in the following sections.
A BCS is called *satisfiable* if there exists a *truth assignment* to all the variables satifying all the constraints. In the example mentioned above, it can be checked that upon adding the four equations, LHS is 0 mod 2, while RHS is 1 mod 2. Hence it is*unsatisfiable.*
A BCS game can be seen as a NL game where the referee sends a constraint c_s to Alice and a variable v_t from the constraint to Bob. Alice then returns the values of variables corresponding to c_s and Bob returns the value for variable v_t. The referee accepts the answers only if:

1. Alices truth assignment satisfies the constraint c_s;
2. Bob's truth assignment for v_t is consistent with Alice's corresponding variable.

B. Satisfying Assignment

As each variable v_j takes value {0,1}, we translate them into variables $V_j = (-1)^{v_j}$ such that they take value {+1,-1}. By quantum satisfying assignment, we mean looking for the set of operators $A_1, A_2 .... A_n$ to the variables $V_1, V_2 .... V_n$ which would have following properties[1]:

1. $A_j{}'s$ are binary observables with eigenvalues $\in$ {+1,1} (i.e.,A_j^2 = I).
2. The order of measurement is irrelevant for a same constraint which means that any pair of observables A_i, A_j corresponding to the same constraint should commute. $[A_i, A_j] = 0$.
3. The operators satisfy each constraint c_s : $\{+1, 1\}^k$ {+1,1} that acts on variables V_1, ..., V_k, such that the equation $c_s(A_1, ..., A_k)$ = -I is satisfied.

C. Perfect Strategy

For classical strategies, determining whether the problem has a perfect strategy is NP-Hard for general BCS games. For *parity* BCS games though, the constraint equations can be reformulated as linear equations and can be solved in polynomial time. Using entanglement however, there is no characterization to look for perfect strategy.
Cleve & Mittal[1] provide a theorem which relates a perfect strategy with the quantum satisfying assignments used earlier.
Theorem : *For any binary constraint system, if there exists a perfect quantum strategy for the corresponding BCS game, then it has a quantum satisfying assignment.*

The detailed *proof* of the theorem can be found in [1]. The idea is that if there exists a perfect strategy, then using a maximally entangled quantum state, the observables can be created using the projection operators of the quantum strategy such that they satisfy the three properties of quantum satisfying assignment. On the other hand, if there exists such an assignement, then all the constraints are satisfied and there exists an observable for every variable. Some of the major points outlining the proof are:

1. The entangled state shared is of the form $|\psi\rangle = \sum_{i=1}^{d} \alpha_i |\phi_i\rangle|\psi_i\rangle$ such that $\alpha_i \geq 0 \vee i$ and $|\phi_i\rangle$ and $|\psi_i\rangle$ are orthonomal vectors.
2. Each constraint c_s has k_s variables. The set of outcomes for Alice are then $\{0, 1\}^{k_s}$. This can be associated with projection operators Π_s^a. For an observable in the constraint, $A_s^j = \sum_{a \in \{0,1\}^{k_s}} (-1)^{a_j} \Pi_a$.
3. The projection operators $A_1, A_2 .... A_n$ are constructed and given a constraint, Alice operates the set observables corresponding to that constraint on her part of the entangled state. Similarly, Bob applies the corresponding observable on his part of the entagled state.

4. Since, the strategy is perfect, they would always agree. i.e. $\langle\psi|A_s^j \otimes B_t|\psi\rangle$ =1 for a variable j in constraint c_s given to Alice and variable v_t on the same constraint given to Bob.

5. Let a same variable be in two constraints c_s and $c_{s'}$ and is denoted as V_s^j and $V_{s'}^{j'}$ respectively. Now since, $\langle\psi|A_s^j \otimes B_t|\psi\rangle = \langle\psi|A_{s'}^{j'} \otimes B_t|\psi\rangle$, this implies $A_s^j = A_{s'}^{j'}$[1]. Hence the observables are non-contextual.

6. Also, since we are dealing with perfect strategy, Alice's output must be consistent with the constraint c_s. That is $\langle\psi|c_s(A_s^1, ..., A_s^j) \otimes B_t|\psi\rangle$ = -1. Hence $c_s(A_s^1, ..., A_s^1) = -I$

D. Condition for No Perfect Strategy

Speelman [12] showed for some BCS games, a simple technique suffices to show the non existence of a quantum satisfying assignment. Taking the previous example we had considered:

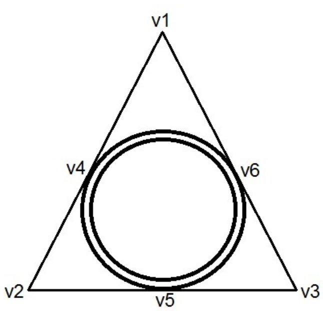

FIG. 3. Pictorial representation of a BCS game

$$v_1 \oplus v_4 \oplus v_2 = 0 \tag{1}$$
$$v_2 \oplus v_5 \oplus v_3 = 0 \tag{2}$$
$$v_3 \oplus v_6 \oplus v_1 = 0 \tag{3}$$
$$v_4 \oplus v_5 \oplus v_6 = 1 \tag{4}$$

We show that this example does not have a quantum satisfying assignment by assuming a contradiction i.e. it has such an assignment. We construct the variables $V_1, V_2 .... V_6$ from the original set $v_1, v_2 .... v_6$ and the corresponding observables are $A_1, A_2 .... A_6$. Then:

$$A_1A_4A_2 = I(first\ constraint)$$
$$A_1A_4A_5A_6A_1 = I(3^{rd}\ constraint A_3 = A_6A_1)$$
$$A_1A_5A_6A_5A_6A_1 = -I(4^{th} constraint A_4 = -A_5A_6)$$
$$A_1A_6A_5A_5A_6A_1 = -I(A_5A_6 = A_6A_5)$$
$$I = -I(Hence\ a\ contradiction)$$

Since this BCS game does not have a quantum satisfying assignment, we will instead try to look for a quantum strategy that wins this BCS game with a probability higher than the classical counterpart.

Classical Strategy and game value:

Let the constraints be denoted by c_1, c_2, c_3 and c_4 respectively.

Constraint	Alice's Strategy		Bob's Strategy	
c_1	0 0	0	0 0	0
c_2	0 0	0	0 0	0
c_3	0 0	0	0 0	0
c_4	0 0	1	0 0	0

TABLE II. The output of the players is depicted according to the constraint and variable sent respectively to Alice and Bob.

This strategy gives the classical winning probability of 11/12. It is seen that the classical game value for this BCS is also $11/12 \approx 0.916$. This is the case because out of 12 possiblities, they will always answer atleast one question incorrectly.

Quantum Strategy:
Alice and Bob decide to share an maximally entagled state $|\psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$. They decide that for inputs v_2 and v_6, their corresponding output would be 0, and they would give same ouput for v_1 and v_5, i.e $v_1 = v_5$. Now their set of constraints look like:

$$v_1 \oplus v_4 = 0 \tag{5}$$
$$v_1 \oplus v_3 = 0 \tag{6}$$
$$v_3 \oplus v_1 = 0 \tag{7}$$
$$v_1 \oplus v_4 = 1 \tag{8}$$

Thus it can be seen that equations (5) & (8) constiute the CHSH inequality that was discussed previously. Equation (6) & (7) are the same.

Constraint	Input	Alice's Strategy	Bob's Strategy
$v_1 \oplus v_4 \oplus v_2$	v_1	CHSH	CHSH
	v_4		CHSH
	v_2		0
$v_2 \oplus v_5 \oplus v_3$	v_2	Equality	0
	v_5		CHSH
	v_3		Equality
$v_3 \oplus v_6 \oplus v_1$	v_3	Equality	Equality
	v_6		0
	v_1		CHSH
$v_4 \oplus v_5 \oplus v_6$	v_4	CHSH	CHSH
	v_5		CHSH
	v_6		0

TABLE III. The table showing a complete quantum strategy for players Alice and Bob.

1. Depending upon the constraint given, Alice either plays with the CHSH operators on her part of the

shared state, or with the Equality operators, or if her constraint has either v_2 or v_6, she outputs the corresponding variable with 0 and plays either CHSH or Equality with the remaining variables in the constraint.

2. Bob also does the same. If his input is either v_2 or v_6, he outputs 0, for v_1, v_4 and v_5, he plays the CHSH game and plays with the Equality operator if he receives v_3.

3. The CHSH operators have been described in the previous XOR section and we know that the probablity of winning the CHSH game is 85%. Thus the players would succeed in scenarios $1^{st}, 2^{nd}, 11^{th}$ and 12^{th} in the table with 85% probablity.

4. The Equality operator is $\mathrm{E} = P_0 - P_1 = |0\rangle\langle 0| - |1\rangle\langle 1|$. Now from the table above, it is clear that there are two out of 12 cases when the players would apply the E operator. The probablity that they would succeed is $\langle\psi|E \otimes E|\psi\rangle = 1$.

5. Thus from the table above, it is clear that in the 6 out 12 scenarios(namely scenarios $3^{rd}, 4^{th}, 6^{th}, 7^{th}, 8^{th}$ and 12^{th}), the players succeed with a probability 1.

6. In the two remaining scenarios, Alice operates with the E operator on her part of the shared state, while Bob operates with the corresponding CHSH operator. Two conditions may arise here with equal probability, either Bob gets the input 0 ir 1.

7. The probability that Alice and Bob give the same output is given by $\frac{1}{2}(\langle\psi|P_0 \otimes B_0^0|\psi\rangle + \langle\psi|P_1 \otimes B_1^0|\psi\rangle + \langle\psi|P_0 \otimes B_0^1|\psi\rangle) + \langle\psi|P_1 \otimes B_1^1|\psi\rangle) = \cos^2(\pi/8) \approx 0.85$.

8. Thus, this quantum strategy yeilds a value of $6/12 + 0.85{*}6/12 = 0.925$

9. Thus we see that this quantum strategy is provides a better winning probability than the best classical strategy.

V. CONCLUSION

The emphasis of this work was to understand non-local games with a class bigger than XOR games. We looked into one special class i.e. BCS games and tried to figure out the conditions under which it would admit a perfect strategy. For XOR games, the quantum game value can be efficiently computed using the semi-definite programming approach, but there is no such technique yet for BCS games. Hence instead of looking for quantum game value, we instead looked for a perfect strategy. We succeeded in specifying some conditions under which a perfect strategy cannot hold. We also looked at a particular example of BCS game where a quantum strategy gives a better value than the classical game value.
A related work could be to figure out an upper bound on the winning probability using a shared quantum state for BCS games. Navascues, Pirinio and Acin [4], talk about a hierarchy of semi-definite programs to test whether a correlation is quantum, and use this idea to give an upper bound on the quantum violation of Bell's Inequalities. This idea could be further developed to talk about the quantum violation of general non-local games as well.

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